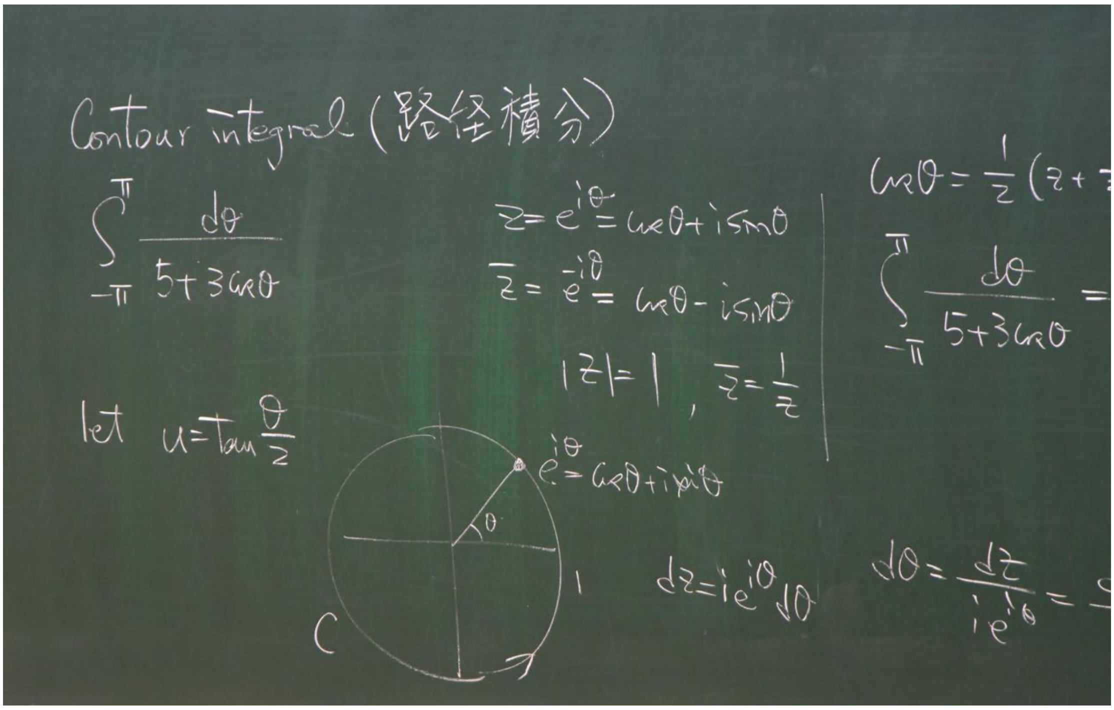
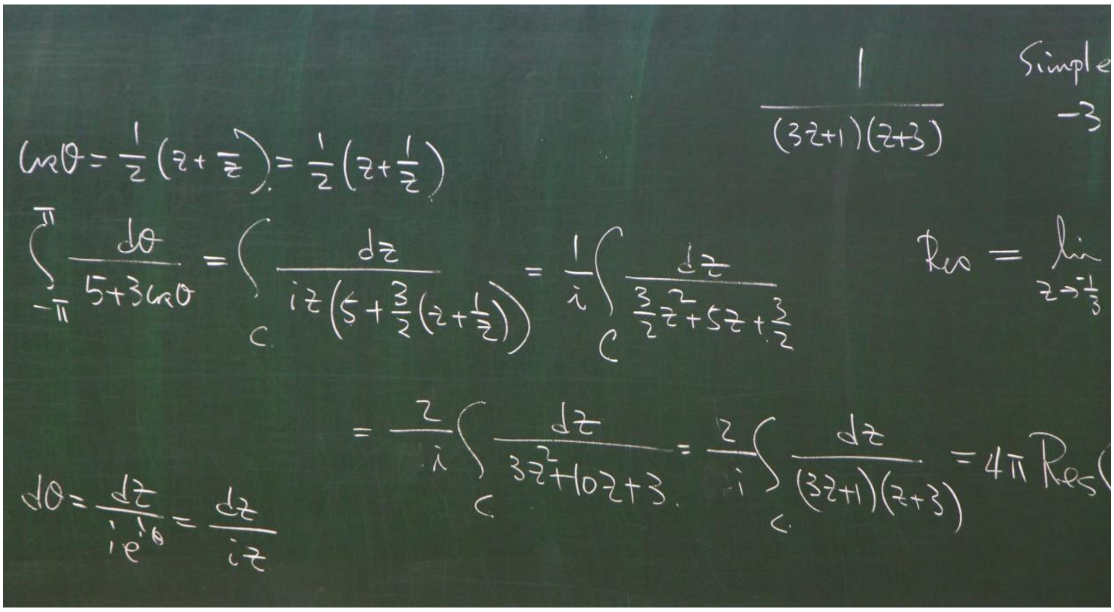
【10920 程守慶教授複變數函數論 / 第 7 堂版書】





$$\frac{1}{(3z+1)(z+3)}$$

Simple poles $-3, -\frac{1}{3}$

$$\int_C \frac{dz}{\frac{3}{2}z^2+5z+\frac{3}{2}}$$

$$\text{Res} = \lim_{z\to -\frac{1}{3}} \left(z+\frac{1}{3}\right)\frac{1}{(3z+1)(z+3)} = \lim_{z\to -\frac{1}{3}} \frac{1}{3z+9} = \frac{1}{8}$$

$$\| $$

$$\ldots\, z+3 = \frac{2}{i}\int_C \frac{dz}{(3z+1)(z+3)} = 4\pi\, \text{Res}\left(\frac{1}{(3z+1)(z+3)}; -\frac{1}{3}\right) = \frac{4\pi}{8} = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx \qquad \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$$\lim_{z\to -\frac{1}{3}} \frac{1}{3z+9} = \frac{1}{8}$$

$$= \frac{\pi}{2}$$

P, Q real-valued polynomials

s.t. $Q(x) \neq 0$, $x \in \mathbb{R}$

$\deg Q \geq 2 + \deg P$.

$$x^4+1 = x^4+2x^2+1-2x^2$$
$$= (x^2+1)^2 - 2x^2$$
$$= (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

$$\frac{P(x)}{Q(x)}\,dx$$

real-valued

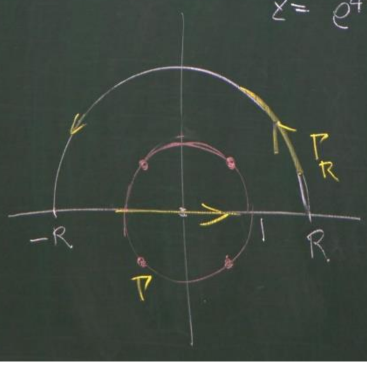
Q polynomials.

$Q(x) \neq 0$, $x \in \mathbb{R}$

$\deg Q \geq 2 + \deg P$.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$$z^4+1=0$$
$$z^4 = -1 = e^{\pi i}$$
$$z = e^{\frac{\pi}{4}i},\ e^{\frac{3}{4}\pi i},\ e^{\frac{5}{4}\pi i},\ e^{\frac{7}{4}\pi i}$$



i, $e^{\frac{5}{4}\pi i}$, $e^{\frac{7}{4}\pi i}$

$R \gg 1$

$$\int_{\Gamma} \frac{dz}{1+z^4} = 2\pi i \,\mathrm{Res}\left(\frac{1}{1+z^4};\ e^{\frac{\pi}{4}i},\ e^{\frac{3}{4}\pi i}\right)$$

$$= 2\pi i\left(\frac{1}{4}\left(-2i\,\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{\sqrt{2}}{2}\pi$$

$$\int_{-R}^{R} \frac{dx}{1+x^4} + \int_{\Gamma_R} \frac{dz}{1+z^4} = \frac{\sqrt{2}}{2}\pi$$

Res. at $e^{\frac{\pi}{4}i} = \lim_{z\to e^{\frac{\pi}{4}i}}$ (

$= \lim_{z\to e^{\frac{\pi}{4}i}}$

Res at $e^{\frac{3}{4}\pi i} = \lim_{z\to e^{\frac{3}{4}\pi i}}$ (

$\frac{1}{1+z^4};\ e^{\frac{\pi}{4}i},\ e^{\frac{3}{4}\pi i}\Big)$

$-2i\,\frac{\sqrt{2}}{2}\ \Big)\Big)$

$\frac{dz}{1+z^4} = \frac{\sqrt{2}}{2}\pi$

Res. at $e^{\frac{\pi}{4}i} = \lim_{z\to e^{\frac{\pi}{4}i}} \left(z - e^{\frac{\pi}{4}i}\right)\frac{1}{z^4+1}$

$$= \lim_{z\to e^{\frac{\pi}{4}i}} \frac{1}{\dfrac{z^4+1-0}{z-e^{\frac{\pi}{4}i}}} = \frac{1}{4e^{\frac{3}{4}\pi i}} = \frac{e^{-\frac{3}{4}\pi i}}{4}$$

$$\frac{\cos\frac{3}{4}\pi - i\sin\frac{3}{4}\pi}{4}$$

Res at $e^{\frac{3}{4}\pi i} = \lim_{z\to e^{\frac{3}{4}\pi i}} \left(z - e^{\frac{3}{4}\pi i}\right)\frac{1}{1+z^4} = \frac{1}{4e^{\frac{9}{4}\pi i}}$

$$= \frac{1}{4e^{\frac{\pi}{4}i}} = \frac{e^{-\frac{\pi}{4}i}}{4}$$

$$\frac{\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}}{4}$$

$\frac{1}{\ (+1)(z+3)}$ Simple -3

Res $= \lim_{z\to}$

$\frac{dz}{\ (+1)(z+3)} = 4\pi\,$Re

$$\lim_{R\to\infty}\left|\int_{\Gamma_R}\frac{dz}{1+z^4}\right|$$

$z = Re^{i\theta}$

$0 \le \theta \le \pi$

$dz = iRe^{i\theta}d\theta$

$$\le \lim_{R\to\infty}\int_0^{\pi}\frac{R\,d\theta}{|1+z^4|}$$

$$\le \lim_{R\to\infty}\int_0^{\pi}\frac{2R\,d\theta}{R^4} = \lim_{R\to\infty}\frac{2\pi}{R^3} = 0$$

$|1+z^4| \ge \frac{|z^4|}{2} = \frac{R^4}{2}$

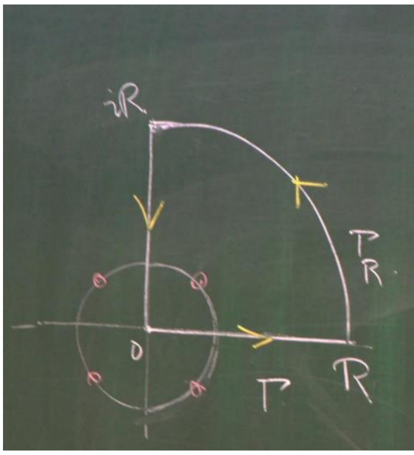
$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$$

real-valu

P, Q polynomia

s.t. $Q(x) \neq 0$. x

$\deg Q \ge 2 + \deg$



$$\int_\Gamma \frac{dz}{1+z^4} = 2\pi i \,\mathrm{Res}\left(\frac{1}{1+z^4};\ e^{\frac{\pi}{4}i}\right) = 2\pi i\left(\frac{e^{-}}{4}\right.$$

$$\int_\Gamma \frac{dz}{1+z^4} = 2\pi i \,\mathrm{Res}\left(\frac{1}{1+z^4};\ e^{\frac{\pi}{4}i}\right) = 2\pi i\left(\frac{e^{-\frac{3}{4}\pi i}}{4}\right) = \frac{2\pi i}{4}\left(\cos\frac{3}{4}\pi - i\sin\frac{3}{4}\pi\right) = \frac{\pi i}{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4}\pi(1-i)$$

$$\int_\Gamma \frac{dz}{1+z^4} = 2\pi i \,\mathrm{Res}\left(\frac{1}{1+z^4};\ e^{\frac{\pi}{4}i}\right) = 2\pi i\left(\frac{e^{-\frac{3}{4}\pi i}}{4}\right) = \frac{2\pi i}{4}\left(\cos\frac{3}{4}\pi - i\sin\frac{3}{4}\pi\right) = \frac{\pi i}{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4}\pi(1-i)$$

$$\| $$

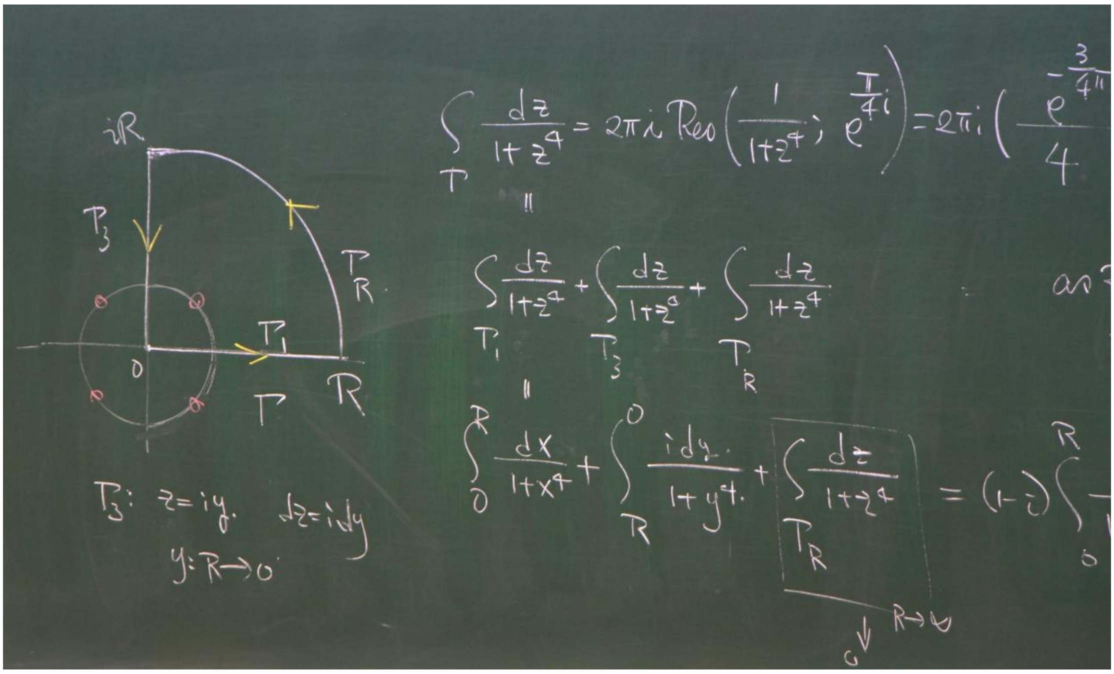
$$\int_{\Gamma_1}\frac{dz}{1+z^4} + \int_{\Gamma_3}\frac{dz}{1+z^4} + \int_{\Gamma_R}\frac{dz}{1+z^4} \qquad \text{as } R\to\infty$$

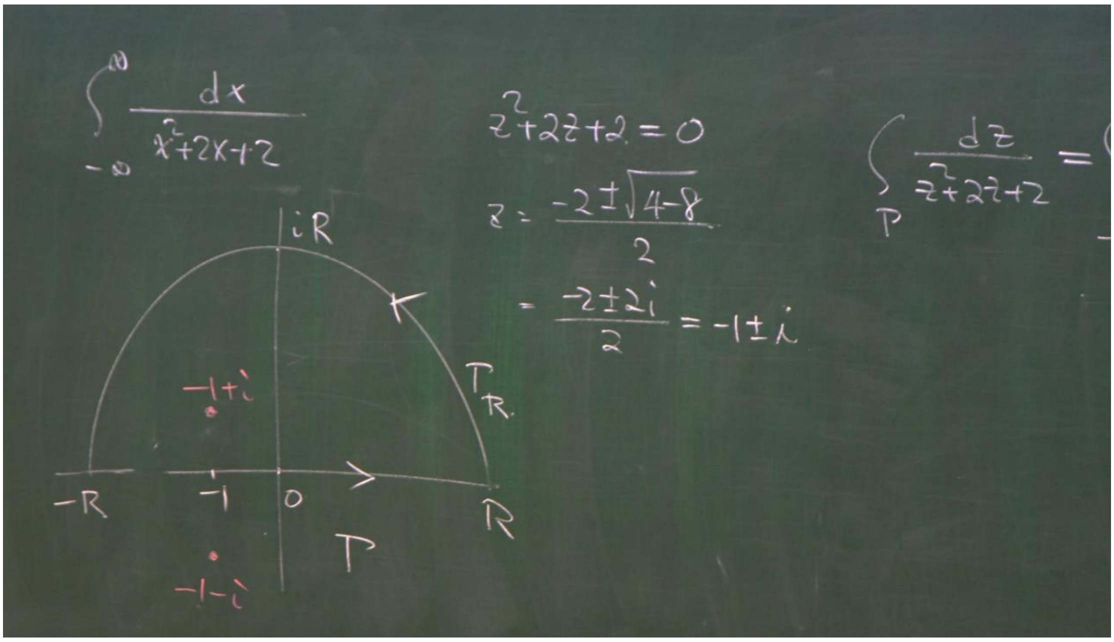
$$\therefore\ (1-i)\int_0^\infty \frac{dx}{1+x^4} = (1-i)\frac{\sqrt{2}}{4}\pi.$$

$$\|$$

$$\int_0^R \frac{dx}{1+x^4} + \int_R^0 \frac{i\,dy}{1+y^4} + \underbrace{\boxed{\int_{\Gamma_R}\frac{dz}{1+z^4}}}_{\xrightarrow{R\to\infty} 0} = (1-i)\int_0^R \frac{dx}{1+x^4} + \int_{\Gamma_R}\frac{dz}{1+z^4}$$

$$\therefore\ \int_0^\infty \frac{dx}{1+x^4} = \frac{\sqrt{2}}{4}\pi.$$


iR
$\int_{\Gamma} \frac{dz}{1+z^4} = 2\pi i \,\mathrm{Res}\left(\frac{1}{1+z^4};\ e^{\frac{\pi}{4}i}\right) = 2\pi i \Big($
$\int_{\Gamma_1} \frac{dz}{1+z^4} + \int_{\Gamma_3} \frac{dz}{1+z^4} + \int_{\Gamma_R} \frac{dz}{1+z^4}$
$\int_0^R \frac{dx}{1+x^4} + \int_R^0 \frac{i\,dy}{1+y^4} + \int_{\Gamma_R} \frac{dz}{1+z^4} = (1-i)\int_0^R$
$\Gamma_3: z = iy.\ dz = i\,dy$
$y: R \to 0$
$R \to \infty$


$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$
$z^2+2z+2=0$
$z = \frac{-2 \pm \sqrt{4-8}}{2}$
$= \frac{-2 \pm 2i}{2} = -1 \pm i$
$\int_{\Gamma} \frac{dz}{z^2+2z+2} =$
iR
$-1+i$
$-1-i$
$-R$
-1
0
R
Γ_R

$$\int_P \frac{dz}{z^2+2z+2} = \int_{-R}^{R} \frac{dx}{x^2+2x+2} + \int_{T_R} \frac{dz}{z^2+2z+2} = 2\pi i \operatorname{Res}\left(\frac{1}{z^2+2z+2}; -1+i\right)$$

$$= \pi.$$

$$\lim_{R\to\infty}\left|\int_{T_R}\frac{dz}{z^2+2z+2}\right| \le \lim_{R\to\infty}\int_0^{\pi} \frac{R\,d\theta}{R^2\left|1+\frac{2}{z}+\frac{2}{z^2}\right|}$$

$z=Re^{i\theta}$

$\theta: 0\to\pi$

$$\le \lim_{R\to\infty}\frac{2\pi}{R}=0$$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}=\pi$$

$$\frac{dz}{z^2+2z+2} = 2\pi i \operatorname{Res}\left(\frac{1}{z^2+2z+2}; -1+i\right) = \pi.$$

$$\int_0^{\pi} \frac{R\,d\theta}{R^2\left|1+\frac{2}{z}+\frac{2}{z^2}\right|} \qquad \lim_{R\to\infty}\frac{2\pi}{R}=0$$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}=\pi$$

$$\lim_{z\to -1+i} \cancel{(z-(-1+i))}\frac{1}{\cancel{(z-(-1+i))}(z-(-1-i))}$$

$$= \frac{1}{(-1+i+1+i)} = \frac{1}{2i}$$

$$\frac{dz}{z^2+2z+2} = 2\pi i \operatorname{Res}\left(\frac{1}{z^2+2z+2}; -1+i\right) = \pi.$$

$$\int_0^{\pi} \frac{R\,d\theta}{R^2\left|1+\frac{2}{z}+\frac{2}{z^2}\right|} \qquad \lim_{R\to\infty}\frac{2\pi}{R}=0$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x-1)(x^2+1)}$$

$$\therefore \int_{-\infty}^{\infty}\frac{x\,dx}{x^2+2x+2}$$

$$\frac{x}{x^2+2x+2} \sim \frac{1}{x}$$

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$$

$$=\lim_{\substack{B\to+\infty\\ A\to-\infty}}\int_{A}^{B}\frac{dx}{1+x^4}$$

$$\int_{a}^{B}\frac{dx}{1+x^4}+\int_{A}^{a}\frac{dx}{1+x^4}.$$

$\underline{S_0=1}$

$1-1+1-1+1-1+\cdots$

Cesaro sum

$$\lim_{n\to\infty}\frac{S_0+S_1+\cdots+S_n}{n+1}=\frac{1}{2}$$

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{(x-1)(x^2+1)}=\lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{1-\varepsilon}+\int_{1+\varepsilon}^{\infty}\right)\frac{dx}{(x-1)(x^2+1)}$$

Principal value 主值

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\,dx}{x^2+2x+2}=\lim_{R\to+\infty}\int_{-R}^{R}\frac{x\,dx}{x^2+2x+2}$$